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Completely integrable equations: dynamical groups and their nonlinear realisations

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Abstract. It is shown that the dynamical groups of completely integrable equations contain as subgroups the infinite groups of symmetry (of the type $G_{n\infty}$), and the infinite Abelian groups of Bäcklund transformations. It is demonstrated that the completely integrable equations possess the elementary Bäcklund transformations of different types. The relation between two methods of describing the completely integrable systems in terms of the field equation and the dynamical group is discussed.

It is also shown that the field theories described by the completely integrable equations are the theories of the Nambu–Goldstone type, i.e. these are the theories of spontaneous breakdown of the symmetry with respect to the infinite dynamical groups, and the corresponding fields are the Goldstone ones by which this spontaneous breakdown is accompanied. In particular, any free field can be considered as a Goldstone one.

1. Introduction

Nonlinear equations describing classical fields are currently attracting considerable interest. The so-called completely integrable equations have been investigated in most detail. A broad class of completely integrable equations studied by the inverse scattering method (see the review papers of Ablowitz *et al* (1974), Scott *et al* (1973) and Flashka and Newell (1975)) possesses a number of interesting properties (solitons, infinite sets of the integrals of motion, etc).

Such specific features of the equations under consideration seem to be associated with the special symmetry properties of these equations. For instance, in a previous paper (Konopelchenko 1977) it has been demonstrated that completely integrable equations have infinite groups of symmetry of the type $G_{n\infty}$.

In the present paper we examine the dynamical groups of completely integrable equations. A dynamical group is the group of transformations converting any solution of the given equation into any other solution of the same equation. We show that the dynamical groups of completely integrable equations contain the infinite groups of symmetry as subgroups (of the type $G_{n\infty}$), as well as the infinite Abelian groups of general Bäcklund transformations. The Bäcklund transformations for different nonlinear equations are extensively studied by Lamb (1971), Wahlquist and Estabrook (1973), Wadati *et al* (1975), Konno and Wadati (1975) and others, and we shall satisfy ourselves that these transformations are included in the dynamical group in a natural way. We shall see also that completely integrable equations contain the Bäcklund transformations of different types. In this paper, it is shown that the property of total

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integrability is closely connected with the fact that the equation has the Abelian group of Bäcklund transformations.

We also show that the field theories described by the completely integrable equations are the theories of the Nambu–Goldstone type; these theories can be considered as those of spontaneous breakdown of infinitely parametric dynamical groups of the symmetry and the corresponding fields as the Goldstone fields by which this spontaneous breakdown is accompanied. In particular, any free field with arbitrary mass and spin can be interpreted as a Goldstone one. Thereby the Goldstone particles can have any values of both mass and spin.

In the second section the dynamical groups of linear equations are considered. The third section is devoted to nonlinear completely integrable equations. The structure of infinite groups of Bäcklund transformations is investigated. The equivalence of two methods for describing the completely integrable systems (by means of the field equation, or the dynamical group) is discussed as well.

In the fourth section it is shown that the theories described by the completely integrable equations result from the Nambu–Goldstone (nonlinear) realisation of the symmetry with respect to some dynamical group. The consequences are discussed in the fifth section.

2. Linear equations

Let us begin our study with the simplest completely integrable equations. Let us consider the field equation

$$\bar{f}(\partial/\partial x)\psi(x)=0 \tag{2.1}$$

where $x=\{x_\mu\}$ are the space–time coordinates ($\mu=0,1,\ldots,N$), $\bar{f}(\partial/\partial x)$ is the arbitrary differential operator, and $\psi(x)$ are the field variables. The space–time dimensionality $N+1$ and the number of field components are arbitrary.

By analogy with the quantum mechanics problems (see for example Kleinert 1968, Aronson *et al* 1974) the group $\mathrm{D}_{\bar{f}}$ of the transformations $\psi\to\psi'$, converting any solution ψ of equation (2.1) into any other solution ψ' of the same equation, will be referred to as a dynamical group of the field equation (2.1). For simplicity, one considers the infinitesimal transformations $\psi\to\psi'=\psi+\delta\psi$. Then, by definition of a dynamical group,

$$\bar{f}(\partial/\partial x)\,\delta\psi(x)=0. \tag{2.2}$$

Taking into account equation (2.1), one finds that $\delta\psi(x)$ is of the form

$$\delta\psi(x)=\sum_\alpha a_\alpha \mathrm{D}_\alpha(x)\omega(x) \tag{2.3}$$

where a_α are the transformation parameters, $\mathrm{D}_\alpha(x)$ are the differential operators commuting with $\bar{f}\{[\bar{f}(\partial/\partial x),\mathrm{D}_\alpha(x)]=0\}$ and $\omega(x)$ is the arbitrary solution of equation (2.1). The transformation of the type (2.3) can be represented in the form of combination of two transformation types:

$$\delta\psi(x)=\sum_\alpha a_\alpha \mathrm{D}_\alpha(x)\psi(x) \tag{2.4}$$

$$\delta\psi(x)=\omega(x). \tag{2.5}$$

The transformations (2.4) are those of the symmetry group of equation (2.1) in the infinitesimal form. In the previous paper (Konopelchenko 1977) it has been shown that the symmetry group of equation (2.1) is the infinite one and belongs to the type $G_{n\infty}$. The arbitrary element of the symmetry group $G_{n\infty}$ is equal to

$$g_{G_{n\infty}} = \exp\Big(\mathrm{i}\sum_\alpha a_\alpha L_\alpha\Big)$$

where $\{L_\alpha, \psi(x)\} = D_\alpha(x)\psi(x)$ and $\{\ ,\ \}$ is the Poisson bracket. The generators L_α of the group $G_{n\infty}$ can be written as

$$L_\alpha = -\int d^N x\, \Pi(x) D_\alpha(x)\psi(x) \tag{2.6}$$

where $\Pi(x)$ are the canonical momenta $\{\psi(\boldsymbol{x}, x_0), \Pi(\boldsymbol{y}, x_0)\} = \delta(\boldsymbol{x}-\boldsymbol{y})$.

Transformations (2.5) also form an infinite group, since $\omega(x)$ is the arbitrary solution of equation (2.1), and the number of independent solutions of such an equation is infinite. The arbitrary element of this group is of the form†

$$g = \exp(\mathrm{i}B_\omega)$$

where

$$B_\omega = -\int d^N x\, \omega(x)\Pi(x) \tag{2.7}$$

and

$$\delta\psi(x) = \{B_\omega, \psi(x)\} = \omega(x). \tag{2.8}$$

The operator B_ω is the generator of transformations (2.5) forming, by virtue of $\{B_\omega, B_{\omega'}\} = 0$, the infinite Abelian group, which will be identified as a group B of Bäcklund transformations. The invariance of equation (2.1) with respect to the group of Bäcklund transformations is, of course, a consequence of the linearity of the equation over the field $\psi(x)$ and the Abelian character of this group (and the corresponding algebra)—the mathematical formulation of the linear superposition principle.

Let us show that the group of Bäcklund transformations may be represented as an infinitely parametric Abelian group. For this, one expands $\omega(x)$ (the arbitrary solution of equation (2.1)) over the total set of solutions for this equation. Let us choose the eigenfunctions of a certain total set of operators Λ_i as a basis in the solution space:

$$\Lambda_i \xi_i(x) = \zeta_i \xi_i(x). \tag{2.9}$$

Writing out $\omega(x)$ in the form

$$\omega(x) = \sum_i \omega_i \xi_i(x)$$

one finds

$$B_\omega = \sum_i \omega_i B_i \tag{2.10}$$

† Note that a gauge group, if equation (2.1) has such a group, is a subgroup of this group. For example, for the vector field A_μ the equation $\Box A_\mu - \partial_\mu\partial_\nu A_\nu = 0$ is invariant with respect to the transformations $A_\mu \to A'_\mu = A_\mu + \omega_\mu$, where $\omega_\mu(x)$ is the arbitrary solution of the first equation. The subgroup of transformations with $\omega_\mu = \partial_\mu \rho(x)$, where $\rho(x)$ is the arbitrary function of coordinates, corresponds to the gauge group.

where

$$B_i = \int d^N x\, \xi_i(x)\Pi(x) \equiv \Pi_i(x). \tag{2.11}$$

Thus, the general Bäcklund transformation is, by virtue of (2.10), the linear superposition of 'elementary' Bäcklund transformations with generators B_i and transformation parameters $\omega_i(-\infty < \omega_i < \infty)$. Since $\{B_i, B_k\} = 0$, this group is Abelian.

For an 'elementary' Bäcklund transformation

$$\delta_k \psi(x) = \omega_k \{B_k, \psi(x)\} = \omega_k \xi_k(x),$$

i.e. the action of the transformation of interest reduces to the addition of the basis element $\xi_k(x)$ with amplitude ω_k, which plays the role of the transformation parameter $(-\infty < \omega_k < \infty)$, to the initial solution $\psi(x)$. With equation (2.9) taken into account, the action of the elementary Bäcklund transformation $\psi(x) \to \psi'(x)$ can also be given as follows:

$$\Lambda_i(x)(\psi'(x) - \psi(x)) = \zeta_i(\psi'(x) - \psi(x)). \tag{2.12}$$

In the papers of Lamb (1971), Wahlquist and Estabrook (1973), Wadati (1975) and others the Bäcklund transformations were considered in just the same form.

One emphasises that, proceding from one basis $\xi_i(x)$ on the solution space to the other, we correspondingly proceed to another form of the 'elementary' Bäcklund transformation. It is evident that the generators of 'elementary' Bäcklund transformations in different bases are connected by linear transformations.

For the translationally invariant equations it is convenient to use the basis consisting of the eigenfunctions of the momentum operator P_μ, i.e. of the plane waves:

$$P_\mu \exp(-ipx) = p_\mu \exp(-ipx).$$

In this basis

$$B_\omega = \int d^{N+1} p \delta(\det \bar{f}(ip))\omega_p B_p \tag{2.13}$$

where

$$B_p = \int d^N x \exp(-ipx)\Pi(x).$$

Correspondingly, for the 'elementary' Bäcklund transformation

$$\delta_p \psi(x) = \omega_p \{B_p, \psi(x)\} = \omega_p \exp(-ipx) \tag{2.14}$$

or

$$i(\partial/\partial x^\mu)(\psi'(x) - \psi(x)) = p_\mu(\psi'(x) - \psi(x)) \tag{2.15}$$

where $\det \bar{f}(ip) = 0$.

Thus the action of a Bäcklund transformation in the momentum basis is reduced to the addition of the plane wave with momentum p and amplitude ω_p, which is the parameter of the transformation $(-\infty < \omega_p < \infty)$, to the solution.

The generators B_i of elementary Bäcklund transformations and the generators L_α of the symmetry group form together the algebra of a dynamical group. The permutation relations for the generators of this algebra can be easily found, following from equations (2.6) and (2.11). The permutation relations for the generators of this symmetry group are given by Konopelchenko (1977). The generators of Bäcklund transformations commute: $\{B_i, B_k\} = 0$. The remaining permutation relations are of the form

$$\{B_i, L_\alpha\} = D_\alpha(\zeta)B_i. \tag{2.16}$$

Here $D_\alpha(\zeta)$ are the infinitesimal operators of the symmetry group in ζ representation, particularly in the momentum representation

$$\{B_q, P_\mu\} = q_\mu B_q.$$

Hence we see that the dynamical group of the linear equation contains transformations of two types: transformations of the symmetry group and the Bäcklund transformation group. The dynamical group acts within the solution space in a transitive manner and, moreover, this is the minimum group for which the solution space of the equation is homogeneous. For a free quantum field all the space of states (the Fock space) is correspondingly the space of the irreducible infinite-dimensional representation of the infinite dynamical group (see Appendix).

3. Nonlinear equations

As is known, a nonlinear equation is completely integrable if it allows such a canonical transformation of the initial canonical variables $\psi(x)$, $\Pi(x)$ to the canonical variables $S(\lambda)$, $\Omega(\lambda, t)$ where the equation under consideration is of the form

$$dS(\lambda)/dt = 0, \qquad d\Omega(\lambda, t)/dt = \delta H\{S\}/\delta S(\lambda) = y_\lambda\{S(\lambda')\}. \tag{3.1}$$

The index λ numerates the infinite set of variables S and Ω, and can take both continuous and discrete values. H is the Hamiltonian.

Since the initial nonlinear equation and equations (3.1) are connected by the canonical transformation, the information contained in one of them coincides with that involved in the other. Particularly, the symmetry groups and the dynamical groups of these equations coincide. Following from this, Konopelchenko (1977) studied the symmetry groups of completely integrable equations. It has been shown that the infinite groups of the type $G_{n\infty}$ are the symmetry groups of such equations.

Here we consider the structure of the dynamical groups of completely integrable equations and, in more detail, the Bäcklund transformations group.

For the sake of simplicity, let us restrict ourselves to the completely integrable equations for which $y_\lambda\{S\} = y_\lambda\{S(\lambda)\}$. Then there is no difficulty in seeing that there exist canonical variables wherein the equations of motion are of the form

$$\partial a(\lambda, t)/\partial t + if(\lambda)a(\lambda, t) = 0 \tag{3.2}$$

where $f(\lambda)$ is a certain function of the index λ.

Equations (3.2) are linear, and repeating the considerations of the foregoing section, we conclude that they possess the infinite group of Bäcklund transformations

$$a(\lambda, t) \to a'(\lambda, t) = a(\lambda, t) + \omega(\lambda, t) \tag{3.3a}$$

or

$$\delta a(\lambda, t) = \{B_\omega, a(\lambda, t)\} = \omega(\lambda, t) \tag{3.3b}$$

where

$$B_\omega = -\int d\lambda\, \omega(\lambda, t) a^*(\lambda, t). \tag{3.4}$$

Here $\omega(\lambda, t)$ is the arbitrary solution of equation (3.2); $a^*(\lambda, t)$ is the quantity canonically conjugate to $a(\lambda, t)$. Since the general solution of equation (3.2) is of the form $\rho(\lambda)\exp(-if(\lambda)t)$, where $\rho(\lambda)$ is an arbitrary function of λ, then for B_ω one obtains

$$B_\omega = \int d\lambda\, \rho(\lambda) B_\lambda \tag{3.5}$$

where

$$B_\lambda = -\exp(-if(\lambda)t) a^*(\lambda, t) = -a^*(\lambda, 0).$$

Thus the general Bäcklund transformation is the superposition of elementary Bäcklund transformations with the generators B_λ, the parameters being the function $\rho(\lambda)$ at fixed λ. Since $\{B_\lambda, B_{\lambda'}\} = 0$ the infinite group of Bäcklund transformations is Abelian. For the elementary Bäcklund transformation,

$$\delta_{\lambda'} a(\lambda, t) = \{\rho(\lambda')B_{\lambda'}, a(\lambda, t)\} = \delta_{\lambda,\lambda'}\rho(\lambda)\exp(-if(\lambda)t), \tag{3.6}$$

and this can be given as follows:

$$(\partial/\partial t)(a'(\lambda, t) - a(\lambda', t)) = if(\lambda)(a'(\lambda, t) - a(\lambda, t)). \tag{3.7}$$

Let us now assume that among the variables $a(\lambda, t)$ there are variables of different types which are numerated correspondingly by indices of different types (continuous, discrete indices). For example, for the equations solved by the inverse scattering method two types of variables a occur; variables of the continuous spectrum (continuous index λ) and variables of the discrete spectrum (soliton variables). For the sine-Gordon equation the discrete variables are divided into two types: the first type of variables corresponds to N soliton solutions, the second to the breathers.

The definite type of 'elementary' Bäcklund transformation corresponds to the definite type of variables. The Bäcklund transformation corresponding to the continuous spectrum adds the solution $\rho(\lambda)\exp(-if(\lambda)t)$ to the initial one. The soliton Bäcklund transformation adds one soliton ($\rho_i \exp(-if_i t)$) to the solution. Similarly to this, the Bäcklund transformation for breathers exists. For each type of the 'elementary' Bäcklund transformations we can write the analogues of formulae (3.6) and (3.7). The generators of different 'elementary' Bäcklund transformations commute. The general Bäcklund transformation is the superposition of 'elementary' Bäcklund transformations of different types:

$$B_\omega = \int d\lambda\, \rho(\lambda) B_\lambda + \sum_i \rho_i B_i + \ldots. \tag{3.8}$$

Since the initial nonlinear completely integrable equation is connected with the equation of the type (3.2) by the canonical transformation, it has the infinite Abelian group of Bäcklund transformations. Existence of the linear principle of superposition for the variables of the type $a(\lambda, t)$ leads to the existence of the nonlinear principle of superposition for the initial field variables $\psi(x)$ (Lamb 1971). The Abelian character of the Bäcklund transformations group is reflected in the commutativity of the corresponding Lamb diagrams (Lamb 1971, Wahlquist and Estabrook 1973, McLaughlin and Scott 1973).

The 'elementary' Bäcklund transformations of different types are simple in the variables $a(\lambda, t)$ (formulae (3.6), (3.7)). They are given in the initial field variables $\psi(x)$ by complex nonlinear transformations whose explicit form can be, as a rule, found only for the variables of the discrete spectrum. This is connected with the fact that the equations of the inverse scattering method can only be solved exactly for the discrete spectrum (soliton variables). For example, for the sine-Gordon equation ($N = 1$)

$$\Box\varphi(x) + (m^3/\sqrt{\lambda})\sin((\sqrt{\lambda}/m)\varphi(x)) = 0 \qquad (\Box \equiv \partial^2/\partial x^\mu \partial x^\mu)$$

the elementary Bäcklund transformation $\varphi \to \varphi'$, corresponding to the addition of one soliton to the solution, is of the following form in the Lorentz-invariant form (for the non-invariant form see for example Lamb 1971):

$$\begin{aligned} \partial\varphi'/\partial x^\mu + \epsilon_{\mu\nu}\,\partial\varphi/\partial x^\nu = {} & (2m^2/\sqrt{\lambda}\,M)p_\mu \cos(\sqrt{\lambda}\,\varphi'/2m)\sin(\sqrt{\lambda}\,\varphi/2m) \\ & - (2m^2/\sqrt{\lambda}\,M)\epsilon_{\mu\nu}p_\nu \sin(\sqrt{\lambda}\,\varphi'/2m)\cos(\sqrt{\lambda}\,\varphi/2m) \end{aligned} \qquad (3.9)$$

where $p_\mu p_\mu = M^2$, M is the soliton mass and $\epsilon_{\mu\nu}$ is the antisymmetric tensor ($\epsilon_{01} = 1$). For other equations integrated by the inverse scattering method, only the soliton elementary Bäcklund transformations are found. However, for these equations there are elementary Bäcklund transformations corresponding to the other parts of the spectrum: continuous and double solitons†.

Hence we see that any nonlinear completely integrable equation has the infinite Abelian group of Bäcklund transformations. One emphasises that the Abelian character of the Bäcklund transformations group is a direct consequence of the total integrability of the equation.

The infinite Bäcklund transformations group and the infinite symmetry group form together an infinite dynamical group of the completely integrable equation. The solution space of this equation is the homogeneous space of the dynamical group. The information of the completely integrable equation coincides with that of its dynamical group.

Thus, the completely integrable field system can be formulated either in terms of a nonlinear equation or a dynamical group. These two formulations give two different but equivalent methods for investigating completely integrable systems.

In this respect, the completely field-integrable systems are infinite (over the number of degrees of freedom); analogues of such systems with finite number of degrees of freedom are the hydrogen atom, the harmonic oscillator, etc (see Kleinert 1968 and Aronson *et al* 1974). The systems indicated above possess an 'obvious' group of symmetry, and also hidden symmetry groups and dynamical groups. These are the finite Lie groups for the systems with a finite number of degrees of freedom (SO(4) and SO(2, 4) for a hydrogen atom; SU(3) and SU(3, 1) for the oscillator), and the infinite Lie groups for the completely integrable systems with an infinite number of degrees of freedom ('hidden' symmetry groups of the type $G_{n\infty}$: infinite dynamical groups). As far as the systems with a finite number of degrees of freedom are concerned, methods of investigation are developed which are based on using the algebraic structure of dynamical groups. Similar methods can also be developed for systems with an infinite number of degrees of freedom, in particular for the completely integrable equations.

† After preparing this paper for publication the author was informed about the papers of Calogero and Degasperis (1976, 1977) in which a large class of Bäcklund transformations was built for certain classes of nonlinear equations. It is possible to show that the Bäcklund transformations of Calogero and Degasperis have the structure described in this section.

4. Completely integrable equations as theories of spontaneous breakdown

In this section we shall show that a theory described by some completely integrable equation can be obtained by the nonlinear (Nambu–Goldstone) realisation of the symmetry under the dynamical group D with the subgroup $G_{n\infty}$ as a vacuum stability group.

Let us first consider the linear equations (2.1). Based upon the explicit form of the generators, it is easy to find the permutation relations of the generators of the group D. One writes those which will be required:

$$[B_p, B_{p1}]_\pm = 0, \qquad [B_q, P_\mu]_- = q_\mu B_q. \tag{4.1}$$

In equation (4.1) the commutator (the minus sign) corresponds to the fields with integer spin, and the anticommutator (the plus sign) corresponds to the fields with half-integer spin. Inclusion of the anticommutator for spinor fields in equation (4.1) is due to the necessity to preserve a correct connection between the spin and statistics. Thus, in the case of fermions, a dynamical group is the supergroup.

Let us now consider the dynamical group D with permutation relations (4.1) as a basic one, forgetting about its origin. We show that the theory described by equation (4.1) arises from the Nambu–Goldstone realisation of the symmetry under the group D, the subgroup $G_{n\infty}$ being the vacuum stability subgroup, and the field $\psi(x)$ turns out to be the Goldstone one.

The nonlinear realisation method developed in detail in the papers of Coleman *et al* (1969), Callan *et al* (1969), Volkov (1973) and Ogievetsky (1974) is the most convenient and natural one for the Nambu–Goldstone realisation of symmetry. According to this method, one must parametrise the quotient space $D/G'_{n\infty}$ (where $G'_{n\infty}$ is the group $G_{n\infty}$ with excluded subgroup of shifts) by means of the fields $\psi_p(x)$ with quantum numbers of the generators B_p:

$$G(x, \psi) = \exp(\mathrm{i}x_\mu P_\mu) \exp\left[\mathrm{i} \int \mathrm{d}^{N+1} \delta(\det \bar{f}(\mathrm{i}p))\psi_p(x) B_p\right]. \tag{4.2}$$

It is further necessary to consider the action of the group D in this quotient space as a group of the left shifts:

$$G(x, \psi) \overset{D}{\rightarrow} G(x, \psi') = g_D G(x, \psi). \tag{4.3}$$

The transformation law of the fields $\psi_p(x)$, as is easy to see, is homogeneous with respect to the group $G_{n\infty}$. Under action of the group B, as follows from equations (4.1)–(4.3), the fields are transformed inhomogeneously. For the elementary Bäcklund transformation with generator B_p we have

$$\delta_p \psi_p(x) = \omega_p \exp(-\mathrm{i}px), \qquad \delta_p \psi_q(x) = 0 \qquad (q \neq p). \tag{4.4}$$

Therefore, the fields $\psi_p(x)$ are fields of the Goldstone type. The field

$$\phi(x) = \int \mathrm{d}^{N+1} p\, \delta(\det \bar{f}(\mathrm{i}p))\psi_p(x) \tag{4.5}$$

covariant under the Lorentz group is also the Goldstone one:

$$\delta_p \phi(x) = \omega_p \exp(-\mathrm{i}px).$$

The Lagrangians invariant with respect to the group D are constructed in a standard way from the fields and their covariant derivatives. Using general prescriptions (see Coleman *et al* 1969, Callan *et al* 1969) for defining the covariant derivatives $\nabla_\mu \psi_p(x)$,

$$G^{-1}(x, \psi)(\partial/\partial x^\mu)G(x, \psi) = iP_\mu + i\int d^{N+1}p\, \delta(\det \bar{f}(ip))\nabla_\mu \psi_p(x) B_p,$$

we find

$$\nabla_\mu \psi_p(x) = (\partial/\partial x^\mu)\psi_p(x) + ip_\mu \psi_p(x). \tag{4.6}$$

Note that the simplicity of the covariant derivative (4.6) is due to choosing in equation (2.1) the basis from the eigenfunctions of the momentum P_μ.

Then, since $\nabla_\mu \psi_p(x)$ and $\Pi_p(x)$ (the canonical momentum conjugate to the field $\psi_p(x)$)† are invariant under the group B, the Lagrangian invariant under the group D contains only $\nabla_\mu \psi_p(x)$ and $\Pi_p(x)$. Thus,

$$\mathscr{L}^{inv}_{(D)} = \mathscr{L}^{inv}_{(G_{n\infty})}(\nabla_\mu \psi_p(x), \Pi_q(x)). \tag{4.7}$$

In the case when the symmetry group $G_{n\infty}$ contains the Lorentz group as a subgroup, $\mathscr{L}^{inv}$ is the integral over the momenta p. Moreover, for those Lagrangians bilinear with respect to the fields $\psi_p(x)$,

$$\mathscr{L}^{inv}_{(D)} = \mathscr{L}^{inv}_{(G_{n\infty})}(\nabla_\mu \psi_p(x), \Pi_q(x)) = \mathscr{L}^{inv}_{(G_n)}(\nabla_\mu \psi_p(x), \Pi_q(x)) \tag{4.8}$$

where G_n is the n parametric symmetry group (see Konopelchenko 1977).

Restricting ourselves now to the Lagrangian bilinear over the fields, we obtain the theory wherein the field $\psi_p(x)$, and thereby the field $\phi(x)$ defined by formula (4.5), satisfy equation (2.1) that permits us to identify the field $\phi(x)$ with the field $\psi(x)$. Let us give two simple examples.

(i) The scalar field with mass m:

$$\begin{aligned}\mathscr{L}^{inv} &= \int d^{N+1}p\, \delta(p^2 - m^2)\nabla_\mu \varphi_p(x)\nabla_\mu \varphi_p(x)\\ &= \int d^{N+1}p\, \delta(p^2 - m^2)[(\partial\varphi_p(x)/\partial x^\mu)\, \partial\varphi_p(x)/\partial x^\mu - m^2\varphi_p^2(x)]\\ &\quad + (\partial/\partial x^\mu)\int d^{N+1}p\, \delta(p^2 - m^2) ip_\mu \varphi_p^2(x).\end{aligned} \tag{4.9}$$

It is seen from equation (4.9) that $\mathscr{L}^{inv}$ is distinguished from the standard Lagrangian for a scalar field only by the term $\partial I_\mu/\partial x^\mu$, where $I_\mu = \int d^{N+1}p\, \delta(p^2 - m^2) ip_\mu \varphi_p^2(x)$, and equations for $\varphi_p(x)$ and $\phi(x)$ are of the following form:

$$(\Box + m^2)\varphi_p(x) = 0, \qquad (\Box + m^2)\phi(x) = 0.$$

(ii) The spinor field with mass m:

$$\begin{aligned}\mathscr{L}^{inv} &= \int d^{N+1}p\, \delta(p^2 - m^2) i\Pi_p(x)\gamma_\mu \nabla_\mu \psi_p(x)\\ &= \int d^{N+1}p\, \delta(p^2 - m^2)\bar{\psi}_p(x)(i\gamma_\mu\, \partial/\partial x^\mu + m)\psi_p(x).\end{aligned} \tag{4.10}$$

† For example, for the Dirac spinor $\Pi_p(x) = \bar{\psi}_p(x) = \psi_p^+(x)\gamma^0$.

The corresponding equations are:

$$(\mathrm{i}\gamma_\mu\,\partial/\partial x^\mu + m)\psi_p(x) = 0, \qquad (\mathrm{i}\gamma_\mu\,\partial/\partial x^\mu + m)\psi(x) = 0.$$

We have thus justified that any field described by a linear equation can be interpreted as a Goldstone one.

Let us proceed now to the general case of nonlinear completely integrable equations. In order to prove the theorem formulated at the beginning of this section, one takes advantage of the fact that a nonlinear completely integrable equation possessing the infinite Abelian group of Bäcklund transformations can be converted into a linear one. After carrying out such a linearising canonical transformation, the considerations mentioned above are applicable to our theory. New canonical variables (wherein the equation is linear)† are the Goldstone ones correspondingly, and can be obtained as a result of the nonlinear realisation of symmetry under the dynamical group D. A structure of the dynamical group (particularly, the Bäcklund transformations group) is invariant with respect to canonical transformations. Nonlinear realisation of the dynamical group which corresponds to initial field variables is connected with nonlinear realisation of the group D in linearised variables by a certain canonical transformation, and therefore this realisation is equivalent to it.‡

In § 3 it has been proved that the information which is involved in the completely integrable equation and that of its dynamical group coincide. The theorem proved by us is indicative of the fact that the completely integrable equation itself can be obtained from the dynamical group by purely group methods.

The fact that completely integrable theories are the theories of spontaneous breakdown indicates also the closeness of the notion of a dynamical group of completely integrable equations (this is a generalisation of the notion of dynamical groups in quantum mechanical problems in the case of an infinite number of degrees of freedom (field)) to the notion of dynamical groups according to Weinberg (1970).

The difference is the following. In the first case, the dynamical groups act within the solution space in a transitive way, i.e. the solution space of the irreducible representations of the dynamical group. In the second case, for example, for the local gauge groups, the solution space is the space of the reducible representation: the solutions with fixed (up to iso-rotations) strength tensor $\mathbf{F}_{\mu\nu}$ form the invariant subspace.

5. On spontaneous breakdown: possible values of quantum numbers for the Goldstone particles

Let us consider the result of the preceding section from the viewpoint of the theory of spontaneously broken symmetries. We have justified that any free field can be interpreted as a Goldstone one. This means that free Goldstone particles can have any values of any quantum numbers. It is clear that this is valid in the general case as well. The values of quantum numbers of the Goldstone particles are determined by the structure of a dynamical group.

In particular, the question about a value of the Goldstone particle's mass is associated, as it is easy to see, with the spectrum of momenta over which the integration for an element of the group B is carried out in expression (2.13). If $\det \bar{f}(\mathrm{i}p) = p^2 - m^2$

† For simplicity, one restricts oneself to the variables of continuous spectrum.

‡ The notion of equivalence of nonlinear realisations is defined in the paper by Coleman *et al* (1969).

and $m \neq 0$, then the Goldstone particle has the mass m†. At $m = 0$ the Goldstone particle is massless. In this case, the group of Bäcklund transformations contains the subgroup of the transformations, which correspond to $p = 0$ (with the generator B_0), for which

$$\delta_0\psi(x) = \omega_0 \tag{5.1}$$

where ω_0 is an arbitrary constant.

Invariance of the equation with respect to the transformation of the form (5.1) is the necessary condition for the massless character of the Goldstone field. In the particular case of the vector field, this result was obtained by Ferrari and Picasso (1971) and Brandt and Wing-Chin (1974).

Usually considered, Goldstone particles are massless. This is connected with the fact that the commutator of the generator corresponds to the spontaneously broken symmetry, or is equal to zero (internal symmetries), or proportional to P^2 (conformal group). The massive Goldstone particle results from the spontaneous breakdown of symmetry whose generators B do not commute with P^2, i.e. $[B, P^2]|\rangle \neq P^2|\rangle$ for all of the states $|\,\rangle$. For instance, for the dynamical group with permutation relations (2.3), $[P^2, B_q] = q_\mu P_\mu B_q + q_\mu B_q P_\mu$ and the states $B_q^+|0\rangle$, where $|0\rangle$ is the vacuum vector ($P_\mu|0\rangle = 0$), describe the particles of mass m:

$$P^2 B_q^+|0\rangle = m^2 B_q^+|0\rangle.$$

Note that, although the cases of massless and massive Goldstone particles are described in the framework of the nonlinear realisation method in the same manner, a physical difference exists between them, namely: in the first case, the non-invariance leads to degeneracy of the states. In particular, the vacuum $|0\rangle$ is degenerate. In the second case, the transformations of spontaneously broken symmetry relate the states with the same action but with different energy, and thereby the vacuum is non-invariant but non-degenerate.

The relation between the values of the mass of the Goldstone field and the structure of transformations of the group B occurs also in the case of non-Abelian groups B, which correspond to nonlinear equations (non-completely integrable ones), for example in the case of non-Abelian gauge groups (the Yang–Mills theory).

Without any variations, the results obtained above are also true for the quantum field theory. In this case, one can use the methods proposed by Dothan and Gal-Ezer (1972). In this paper, a principal possibility of the existence of massive Goldstone particles has been apparently noted for the first time.

Appendix

Here we shall show that the space of states of the completely integrable theory is the space of one irreducible infinite-dimensional representation of the group D.

Let us consider linear equations (§ 2). The permutation relations of the algebra of the dynamical group D are of the form (in momentum representation):

$$[P_\mu, P_\nu]_- = 0, \qquad [B_q, P_\mu]_- = q_\mu B_q, \qquad [B_p, B_{p1}]_\pm = 0 \tag{A.1}$$

plus the permutation relation of the remaining generators of the group $G_{n\infty}$ with the generators B_p, which will not be written out (see equation (2.16)).

† The neutral massive vector field theory belongs to this case.

To construct the representations of a dynamical group, it is necessary, as usual, to choose the total set of commuting generators. The general eigenfunctions of these generators will be the basis functions, and the eigenvalues of these generators will numerate the basis. It is seen from the permutation relations that, in the case of the fields with integer spin, there exist at least two such sets. The first consists of the operators of the momentum P_μ and the generators of the symmetry group $G_{n\infty}$, which commute with P_μ, i.e. of the generators of the maximum Abelian subgroup of the group G_n. The second set consists of the generators B_p of elementary Bäcklund transformations. For fermions there exists only the first possibility.

For simplicity let us restrict ourselves to the scalar field theory. Let us consider now the basis from the eigenfunctions of the momentum

$$P_\mu|p\rangle = p_\mu|p\rangle. \tag{A.2}$$

From the relations (A.2) it follows that

$$P_\mu B_q^+|p\rangle = (p_\mu + q_\mu)B_q^+|p\rangle, \qquad P_\mu B_q|p\rangle = (p_\mu - q_\mu)B_q|p\rangle,$$
$$P^2 B_q^+|0\rangle = m^2 B_q^+|0\rangle. \tag{A.3}$$

Suppose that there exists the vector $|0\rangle$, so that

$$P_\mu|0\rangle = 0. \tag{A.4}$$

Then there is no difficulty in seeing that the vectors

$$|0\rangle,\ B_p^+|0\rangle,\ B_p|0\rangle,\ B_{p_1}^+B_{p_2}^+|0\rangle,\ B_{p_1}B_{p_2}^+|0\rangle,\ B_{p_1}^+B_{p_2}|0\rangle,\ B_{p_1}B_{p_2}|0\rangle\ \ldots \tag{A.5}$$

form the basis of the irreducible infinite-dimensional representation of the group D. In the case of Lorentz-invariant theories we can remove negative energies, extracting positive- and negative-frequency parts from the generators B_p and performing the usual reinterpretation. A similar construction is true for the case of an arbitrary free field.

Thus, the space of an irreducible infinite-dimensional representation of a dynamical group is of the form

$$|0\rangle \oplus B_p^+|0\rangle \oplus B_p|0\rangle \oplus B_{p_1}^+|0\rangle \otimes B_{p_2}^+|0\rangle \oplus \ldots. \tag{A.6}$$

It is easy to see that the construction (A.6) coincides with the construction, which is well known in the quantum field theory, of Fock space, i.e. the space of states of free quantum field theory (see for example Schweber 1961).

If we choose the eigenfunctions of the operators B_p as a basis, then, taking into account that B_p is a linear combination of the operators of creation and annihilation, one obtains the so-called coherent representation.

The conclusions made above are valid also for nonlinear completely integrable equations, if one considers them in linearising variables. Then only the operators B_i corresponding to the discrete spectrum are added. They have the sense of the operators of creation and annihilation of the solitons. However, in transition to the initial field variables, difficulties arise due to ambiguity of canonical transformations in the quantum case.

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